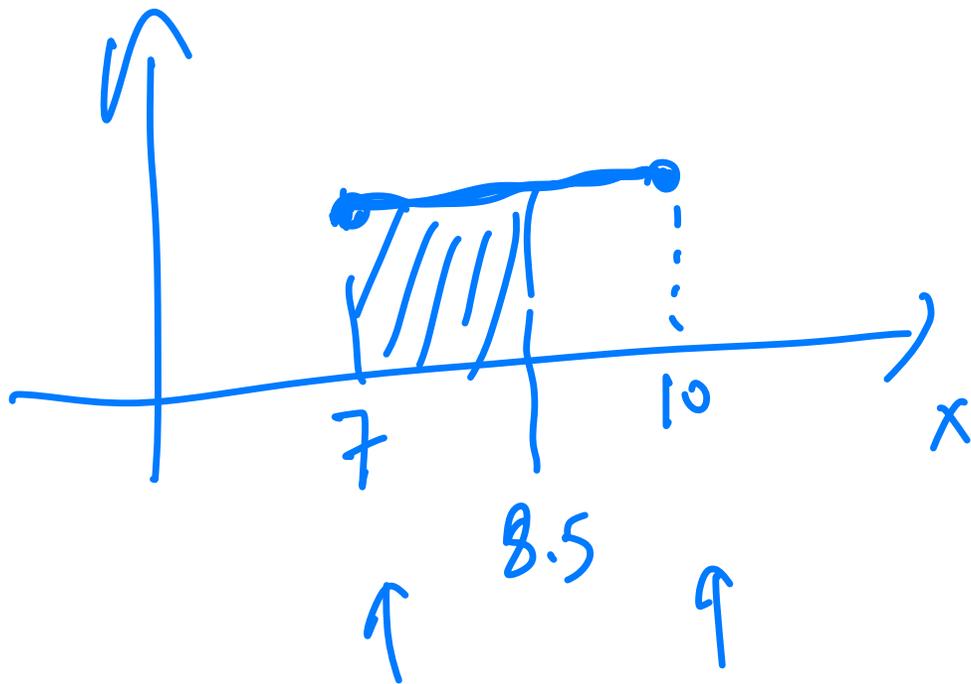


# QUIZ

The daily amount of coffee, in liters, dispensed by a machine is a random variable  $X$  having a **continuous uniform distribution** with  $A = 7$  and  $B = 10$ . Find the probability that on a given day the amount of coffee dispensed by it will be  $\leq 8.5$  liters.

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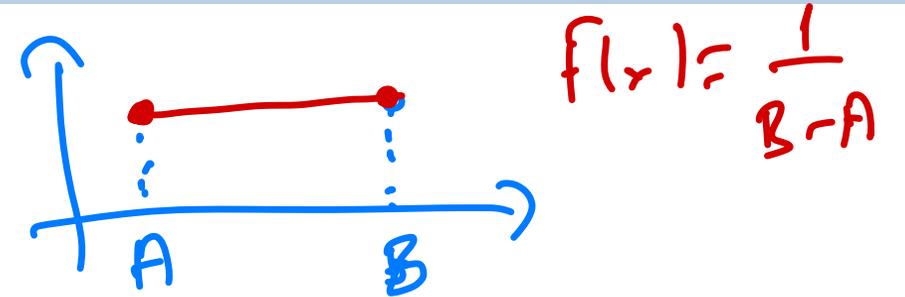
$$\begin{aligned} P(X \leq 8.5) &= \int_7^{8.5} \frac{1}{3} dx \\ &= \left. \frac{x}{3} \right|_7^{8.5} \\ &= 0.5 \end{aligned}$$

# Summary of Previous Lecture

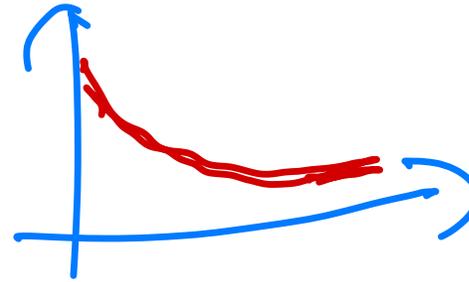
- Continuous Uniform Distribution
- Exponential Distribution
- Gamma Distribution
- Normal Distribution
  - Area under the normal curve
  - Normal approximation of the binomial
- Lognormal Distribution

# Summary of Previous Lecture

➤ Continuous Uniform Distribution



➤ Exponential Distribution



$$f(x) = \lambda \cdot e^{-\lambda x}$$

➤ Gamma Distribution

$$\text{mean } \frac{1}{\lambda}$$

➤ Normal Distribution

➤ Area under the normal curve

➤ Normal approximation of the binomial

➤ Lognormal Distribution

# Overview

- Continuous Uniform Distribution
- Exponential Distribution (See some examples)
- Gamma Distribution (Skip)
- Normal Distribution
  - Area under the normal curve
  - Normal approximation of the binomial
- Lognormal Distribution

# Some Exercises on Exponential Distribution

**Example 6.17:** Suppose that a system contains a certain type of component whose time, in years, to failure is given by  $X$ . The random variable  $X$  is modeled nicely by the exponential distribution with mean time to failure  $\mu = 5$ . If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years? Use  $e^{-8/5} \approx 0.2$  and

```
print (1-binom.cdf(1, 5, 0.2))  
0.26272000000000006
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**Solution:**

# Some Exercises on Exponential Distribution

**Example 6.17:** Suppose that a system contains a certain type of component whose time, in years, to failure is given by  $X$ . The random variable  $X$  is modeled nicely by the exponential distribution with mean time to failure  $= 5$ . If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

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Solution:

$$P(X > 8) = e^{-1.6} = e^{-\frac{8}{5}} = 0.2$$

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$y_1, y_2, y_3, y_4, y_5$

↓  
success(1)  
if component 1 lasts  $> 8$  years.  
with  $p = 0.2$

success  
if component 5 lasts  $> 8$  years  
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Solution:

$$P(X > 8) = e^{-\lambda \cdot 8} = e^{-\frac{8}{5}} = 0.2$$

$Y_1, Y_2, Y_3, Y_4, Y_5$

✓  
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if component 1 lasts  $> 8$  years.  
with  $p = 0.2$

$Y$ : Binomial distribution

$$n = 5, p = 0.2$$

$$P(Y > 2) = 1 - P(Y \leq 1) = 0.26\dots$$

# Some Exercises on Exponential Distribution

**Example:** An exam is taken by 5 students independently. The number of minutes required by any particular student to complete it has exponential distribution with mean 80. The exam starts at 9:00 a.m. Probability that at least one of the students will complete the exam before 9:40 a.m.?

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Probability  $\geq 1$  student finish in  $\leq 40$  mins =  $1 -$  probability that no one finishes in  $\leq 40$  mins

$$= 1 - \left(e^{-\frac{1}{2}}\right)^5 = 1 - e^{-\frac{5}{2}}$$

# Outline

- Continuous Uniform Distribution
- Exponential Distribution
- Gamma Distribution
- **Normal Distribution**
  - **Area under the normal curve**
  - Normal approximation of the binomial
- Lognormal Distribution

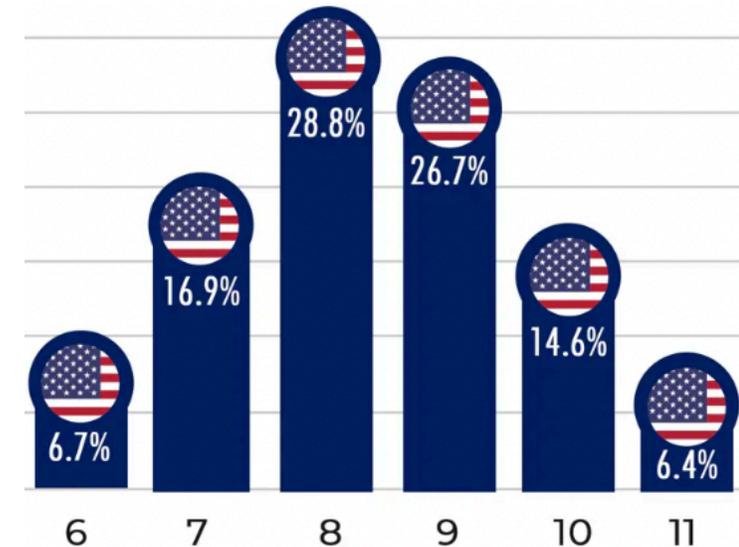
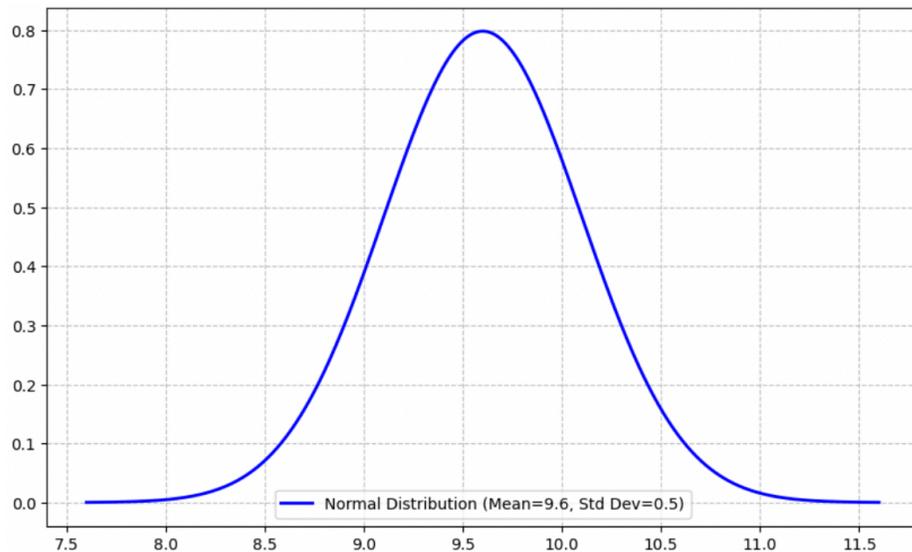
# Normal Distribution

## Example: Charlie and the Shoe Factory

Charlie has a shoe factory for making women's shoes. The factory can produce 1200 shoes per year. How many shoes from each size should Charlie produce? Say 6 different sizes:  $1200/6=200$  from each size?



If he knew the distribution for women's feet sizes?



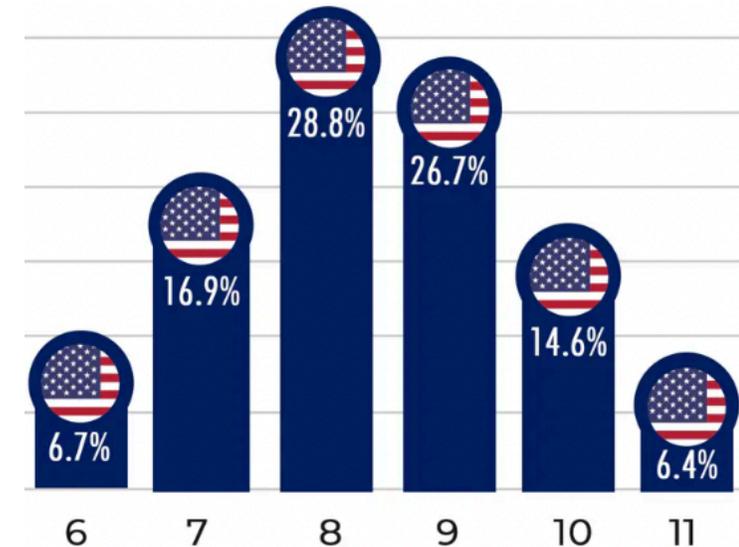
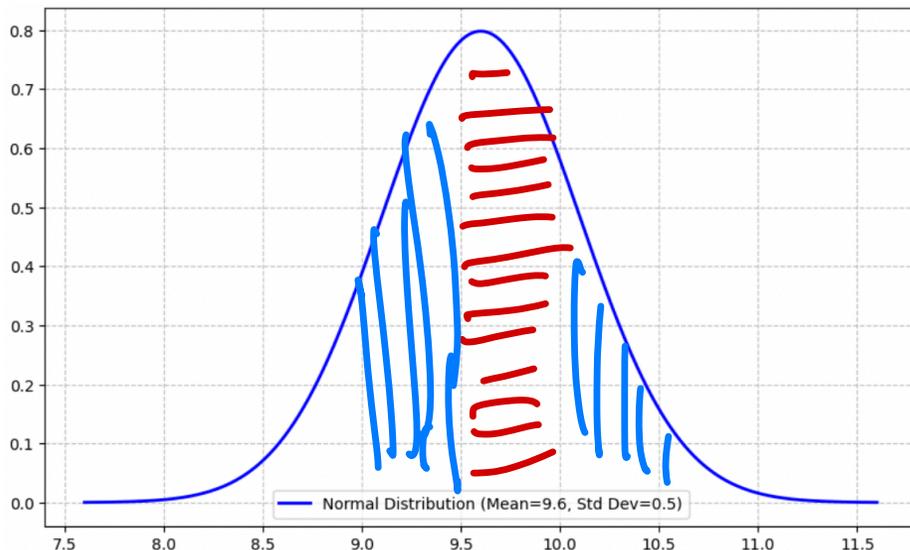
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**Definition:**  $X$  has Normal (Gaussian) distribution with parameters  $\mu$ ,  $\sigma^2$ , if

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

Denoted with  $X \sim N(\mu, \sigma^2)$

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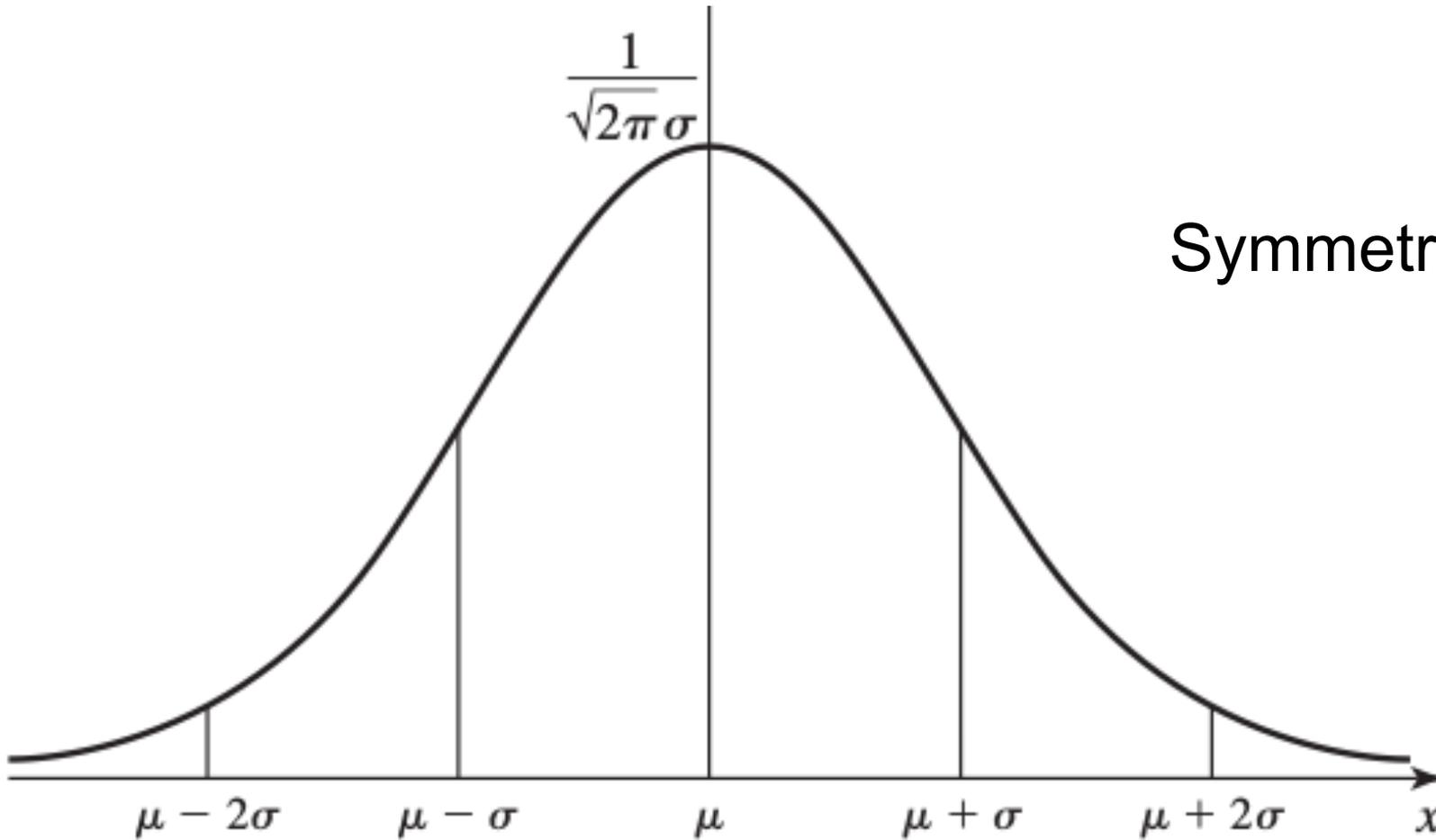
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A very common distribution:

Many physical phenomena are approximately normal

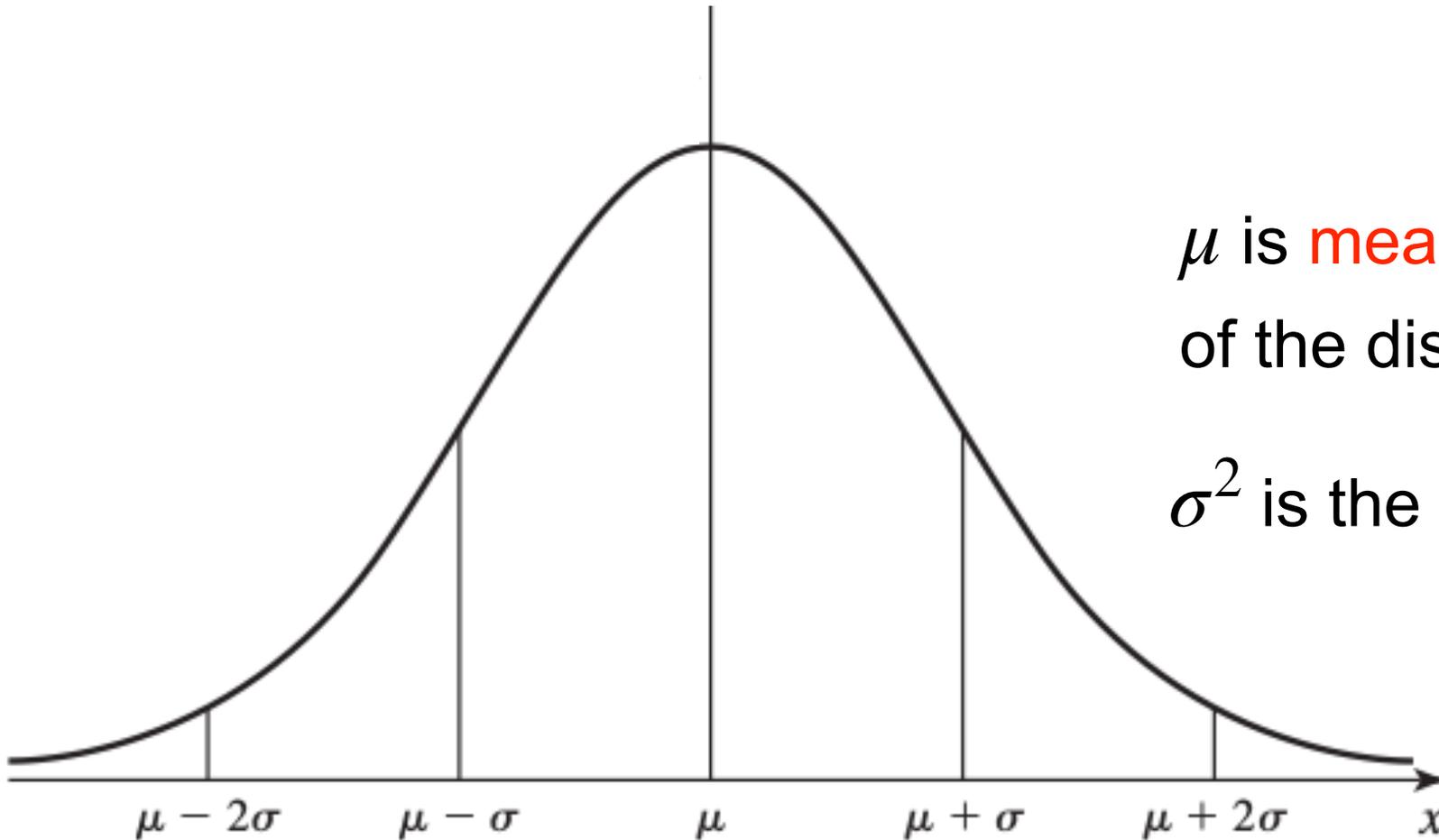
Central Limit Theorem: Mean of samples from **any distribution** is approximately normal (We'll see the details later.)

# Normal Distribution



Symmetric around  $\mu$ .

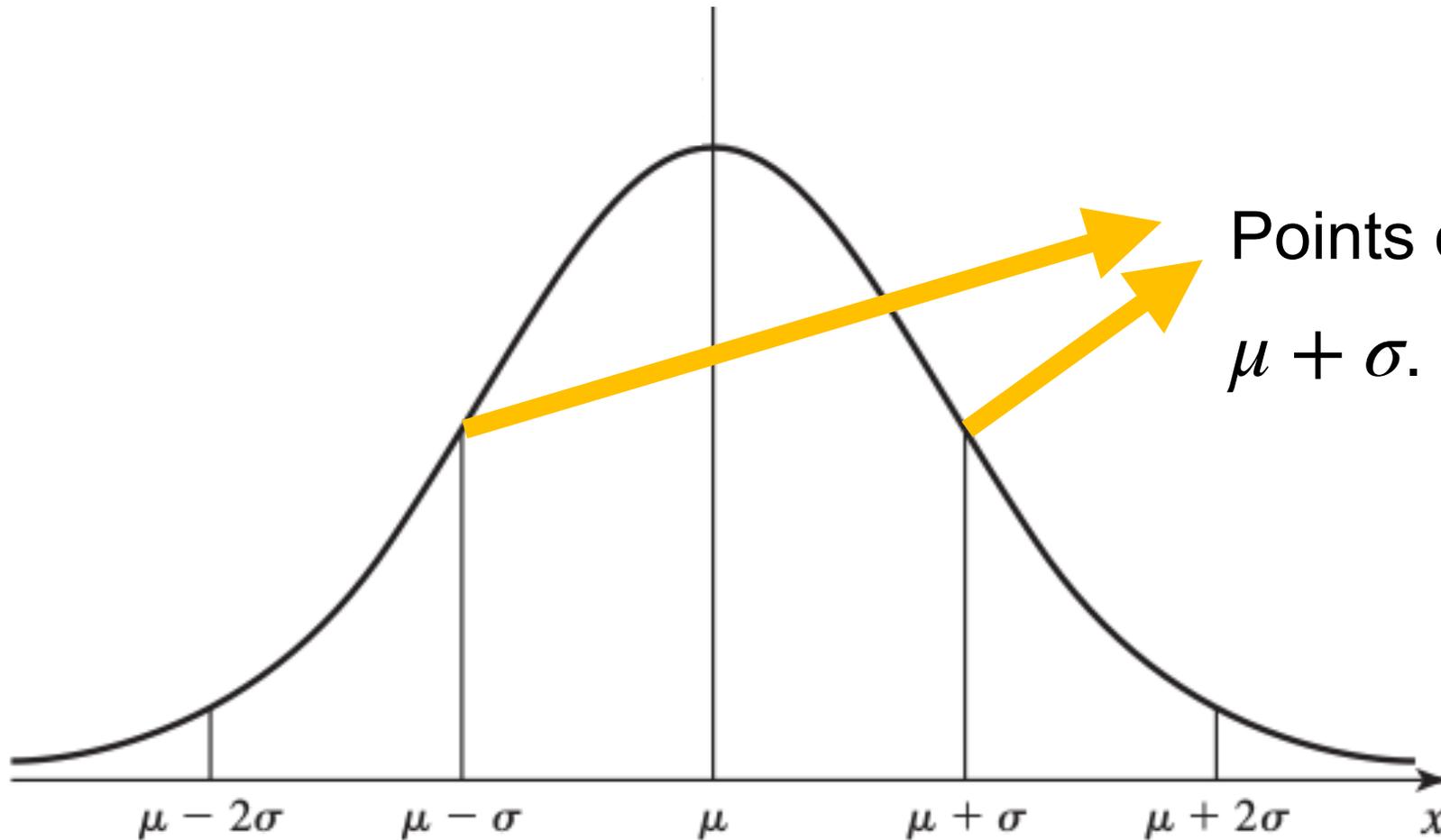
# Normal Distribution



$\mu$  is **mean**, **median**, and the **mode** of the distribution.

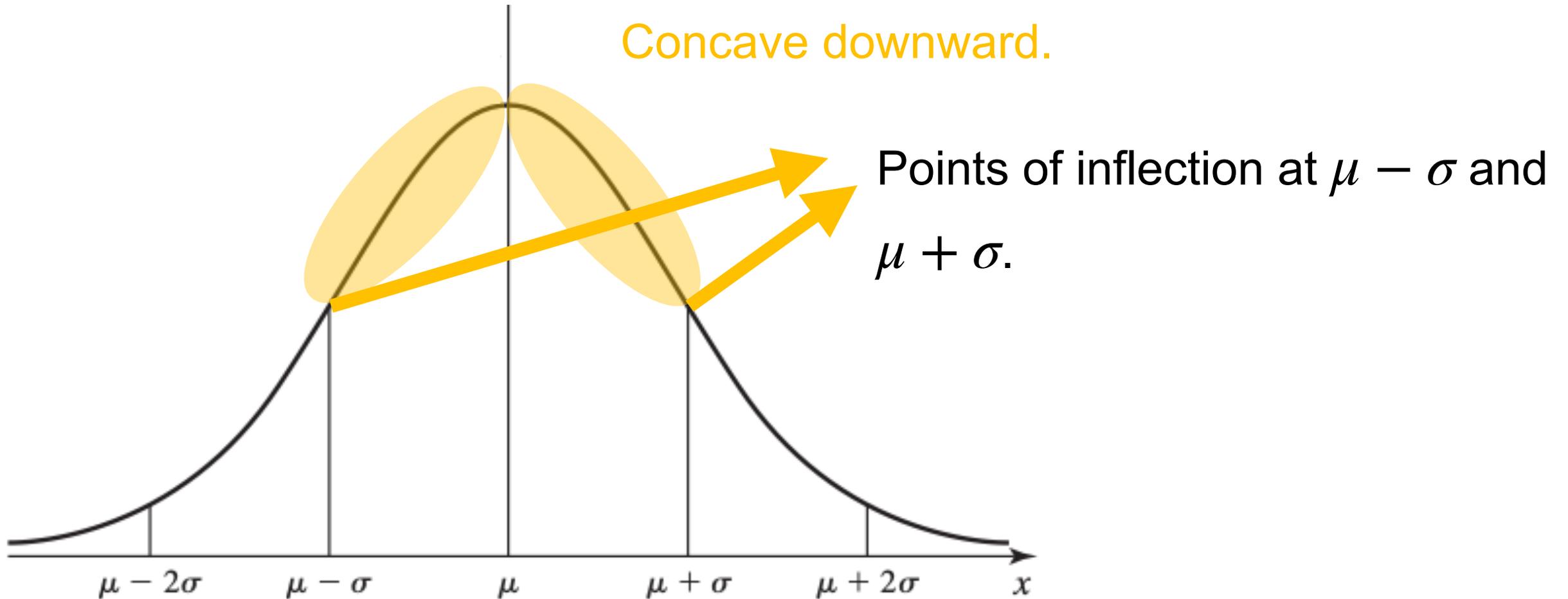
$\sigma^2$  is the **variance** of the distribution.

# Normal Distribution

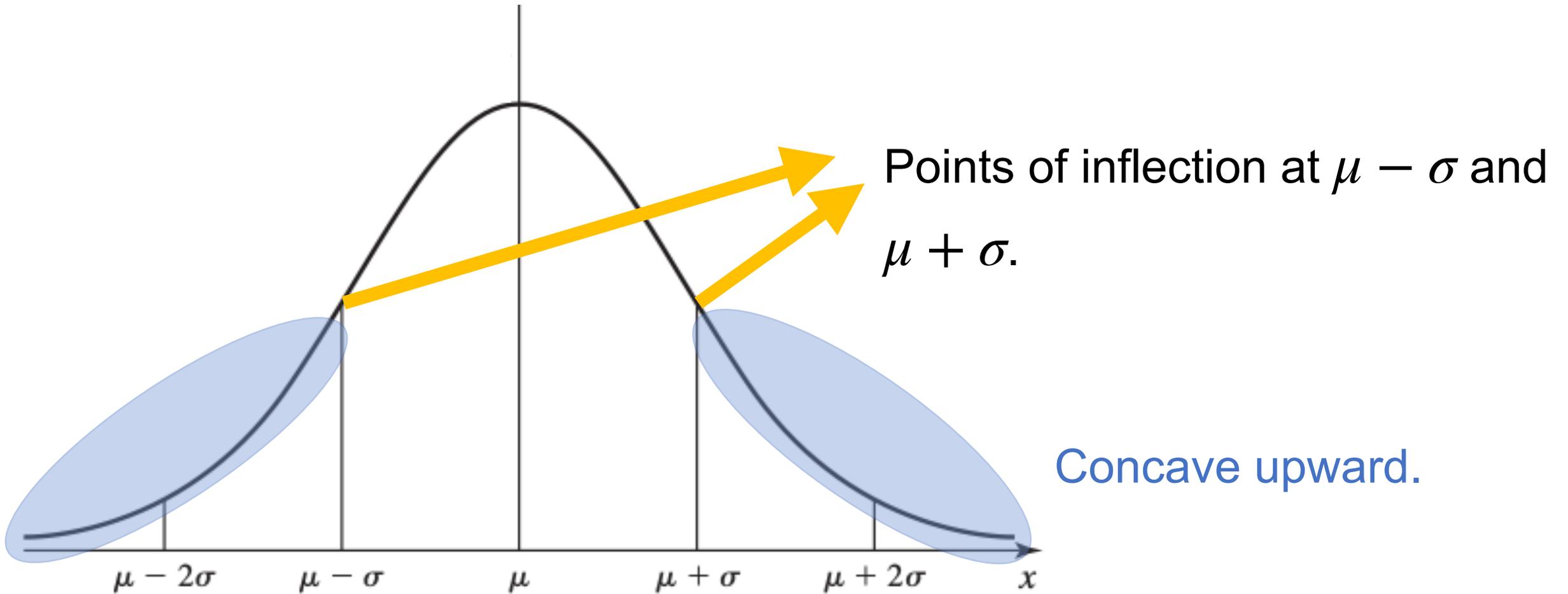


Points of inflection at  $\mu - \sigma$  and  $\mu + \sigma$ .

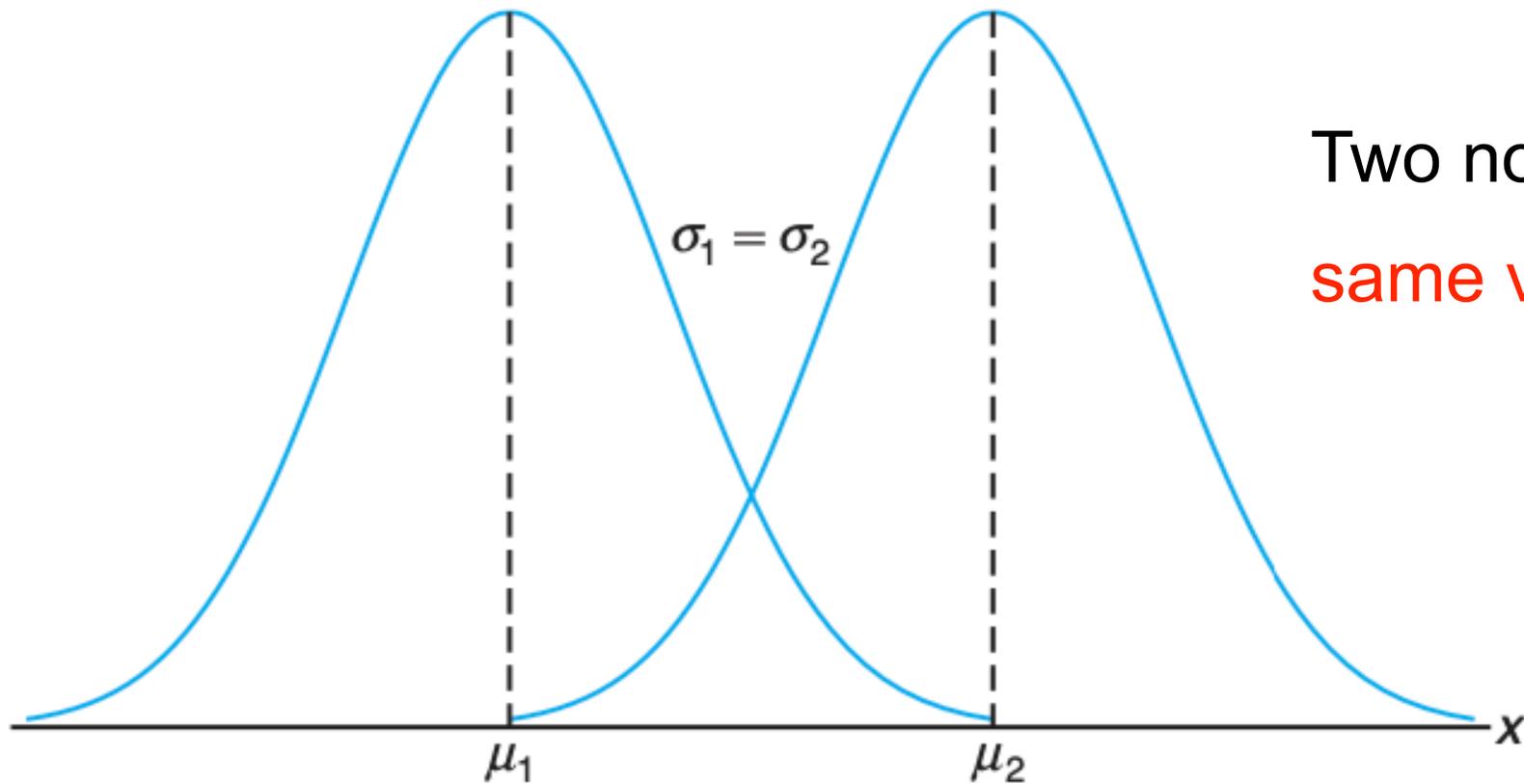
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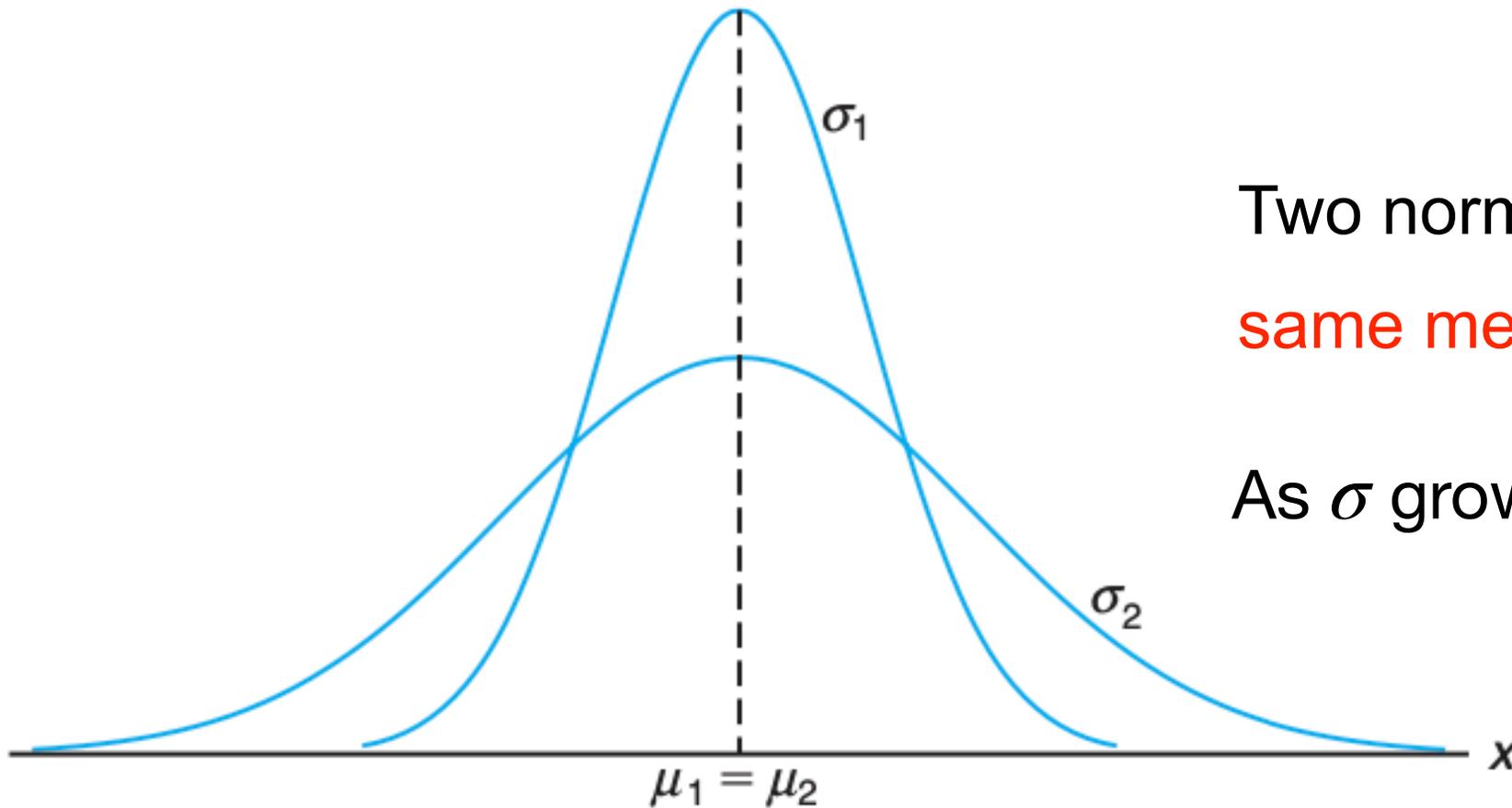


# Normal Distribution



Two normal distributions with the same variance and  $\mu_1 < \mu_2$ .

# Normal Distribution



Two normal distributions with the same mean and  $\sigma_1 < \sigma_2$ .

As  $\sigma$  grows, pdf gets more flat.

# Normal Distribution

Linear combinations/transformations of normal independent variables:

If  $X_1, X_2$  are independent and each is normally distributed then

$Y = a_1X_1 + a_2X_2 + b$  has a normal distribution.

(Can be generalized to more than 2 random variables.)

**Revisit:** What is the mean and variance of  $Y$ ?

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(Linearity of expectation)

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Remember

$$\begin{aligned}\text{Var}(ax) &= \\ E[(ax - E[ax])^2] &= \\ E[(ax - aE[x])^2] &= \\ E[a^2(x - E[x])^2] &= \\ a^2 \text{Var}(x) &= \end{aligned}$$

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and  $\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2)$   
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# Normal Distribution

Generalizes to sum of independent random variables (regardless of their distributions)

More on this later when we talk about the central limit theorem.