# CSC196: Analyzing Data

## Continuous Random Variables

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# Outline

- Concepts of Calculus
- Continuous Probability Distributions
- Fundamental Rules of Probability

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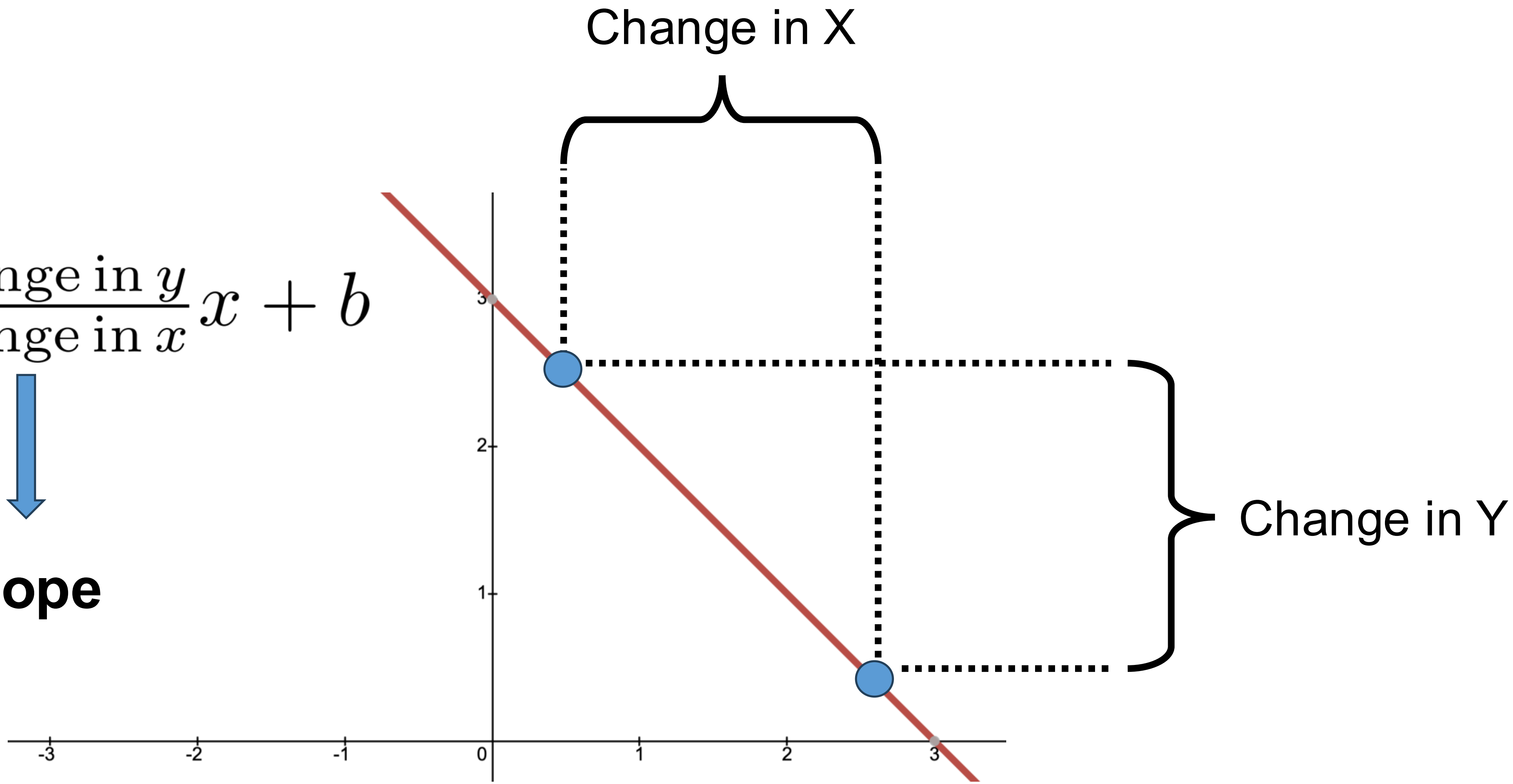
# Equation for a Line

*Recall the equation for a line in point / slope form…*

$$y = \frac{\text{Change in } y}{\text{Change in } x} x + b$$

**Slope**

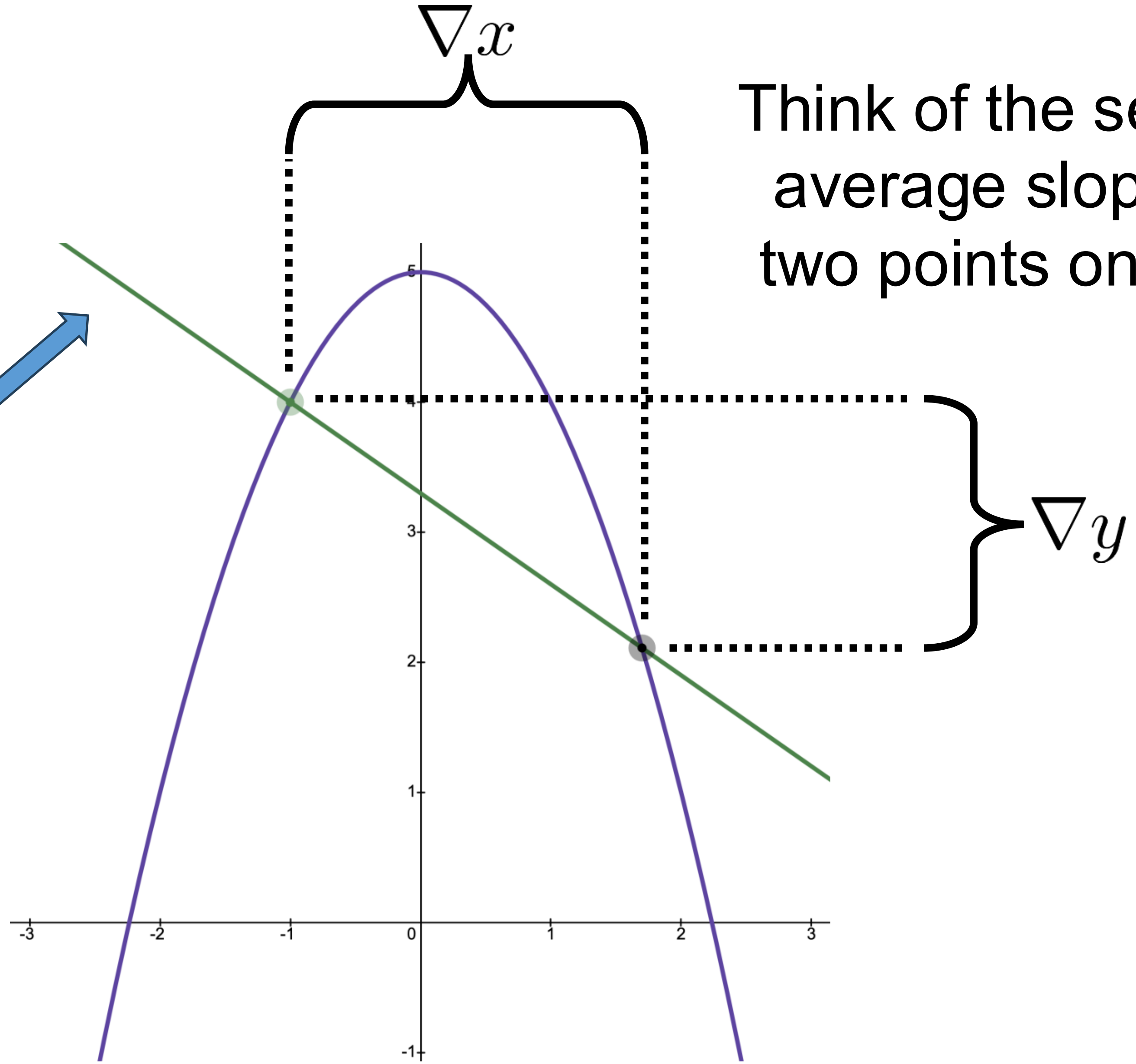


# Secant

Think of the secant as the average slope between two points on a function.

$$y = \frac{\nabla y}{\nabla x} x + b$$

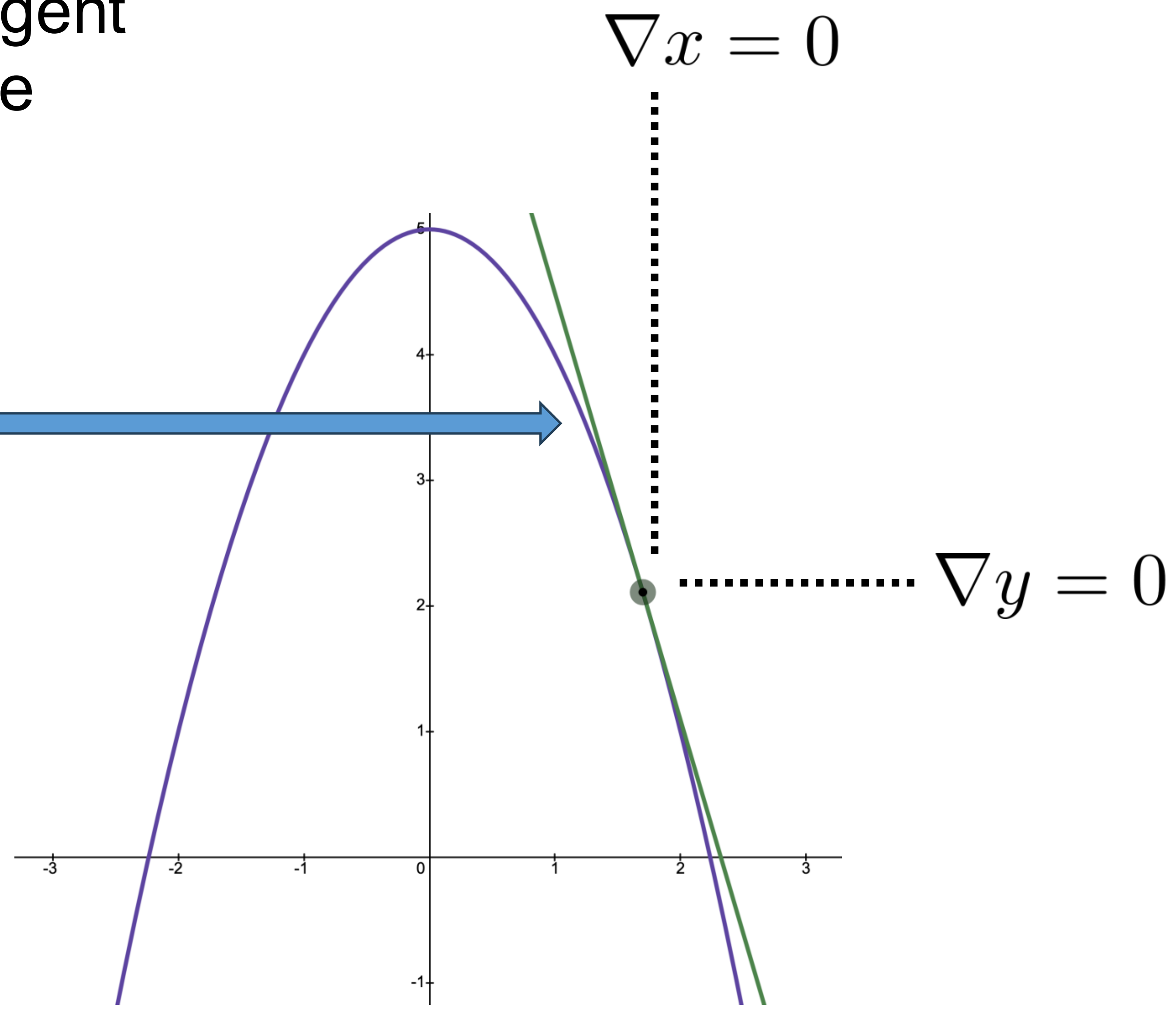
**Slope**



# Tangent

The slope of the tangent line appears to be undefined…

$$y = \frac{0}{0}x + b$$

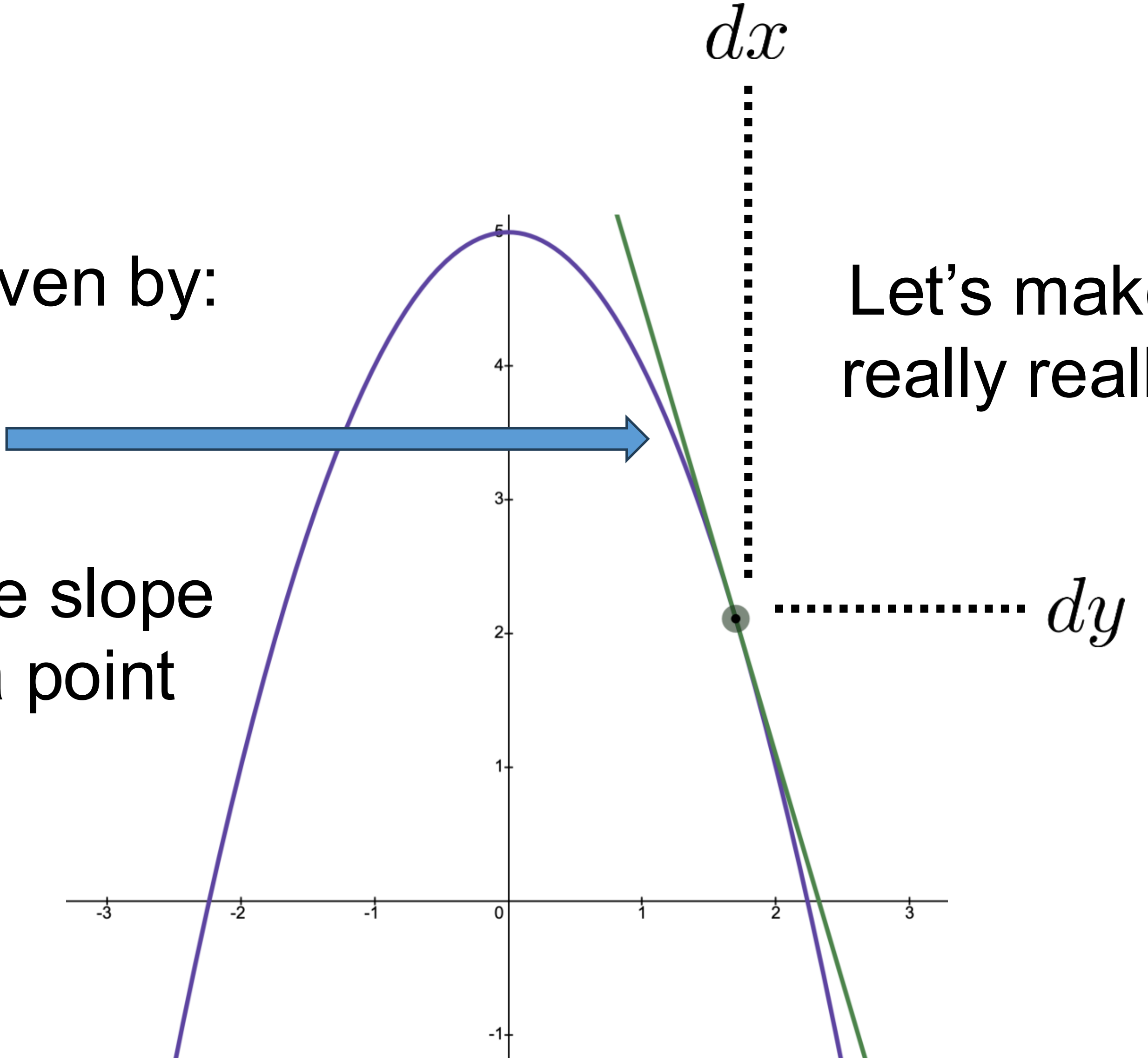


# Derivative

The slope is now given by:

$$\frac{dy}{dx}$$

The *derivative* is the slope of the tangent at a point




Let's make the difference really really really small…

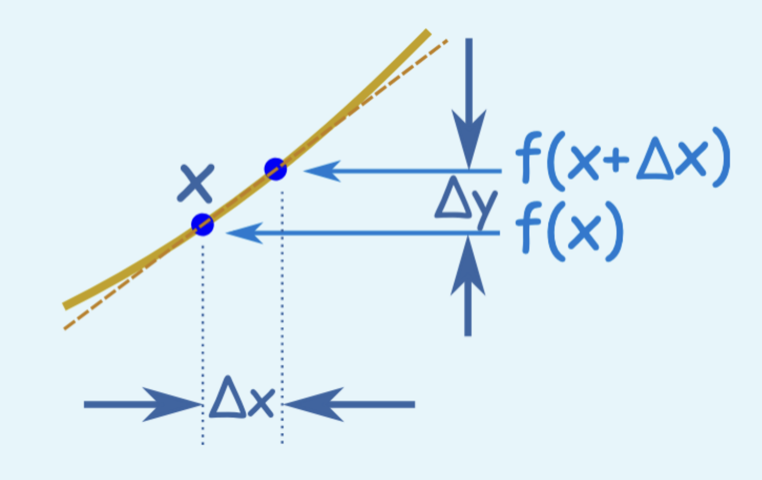
# How to Calculate a Derivative

To find the derivative of a function $y = f(x)$ we use the slope form:

$$\text{Slope} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\nabla y}{\nabla x}$$

X changes from: $x \to x + \nabla x$

Y changes from: $f(x) \to f(x + \nabla x)$




Start with the slope form: $\frac{\nabla y}{\nabla x} = \frac{f(x+\nabla x) - f(x)}{\nabla x}$

Simplify, and make $\nabla x$ shrink towards zero…

# Example: Derivative

Example: the function $\mathbf{f(x) = x^2}$

The slope formula is: $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Use $f(x) = x^2$: $\dfrac{(x+\Delta x)^2 - x^2}{\Delta x}$

Expand $(x+\Delta x)^2$ to $x^2+2x\,\Delta x+(\Delta x)^2$: $\dfrac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$

Simplify ($x^2$ and $-x^2$ cancel): $\dfrac{2x\,\Delta x + (\Delta x)^2}{\Delta x}$

Simplify more (divide through by $\Delta x$): $2x + \Delta x$

Then, **as $\Delta x$ heads towards 0** we get: $2x$

Result: the derivative of $\mathbf{x^2}$ is $\mathbf{2x}$

In other words, the slope at x is **2x**

Source: mathisfun.com

# Derivative

Instead of saying “$\nabla x$ heads towards zero” we write ”dx” so we have:

$$\frac{d}{dx}x^2 = 2x$$

But what does this actually mean?

- For function x2 the slope (or rate of change) at any point is 2x
- So if x=2 then the slope is 2x=4
- If x=5 then the slope is 2x=10
- And so on…

# Another Derivative Example

Example: What is $\frac{d}{dx}x^3$ ?

We know $\mathbf{f(x) = x^3}$, and can calculate $\mathbf{f(x+\Delta x)}$, so let's go:

The slope formula: $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

Use $f(x) = x^3$: $\frac{(x+\Delta x)^3 - x^3}{\Delta x}$

Use $(x+\Delta x)^3 = x^3 + 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3$

Replace $(x+\Delta x)^3$: $\frac{x^3 + 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

Simplify ($x^3$ and $-x^3$ cancel): $\frac{3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3}{\Delta x}$

Simplify more (divide through by $\Delta x$): $3x^2 + 3x\,\Delta x + (\Delta x)^2$

As $\Delta x$ heads towards 0 we get: $3x^2$

Result: the derivative of $\mathbf{x^3}$ is $\mathbf{3x^2}$

Source: mathisfun.com

# Derivative Rules

*In practice it is easier to memorize derivative rules…*

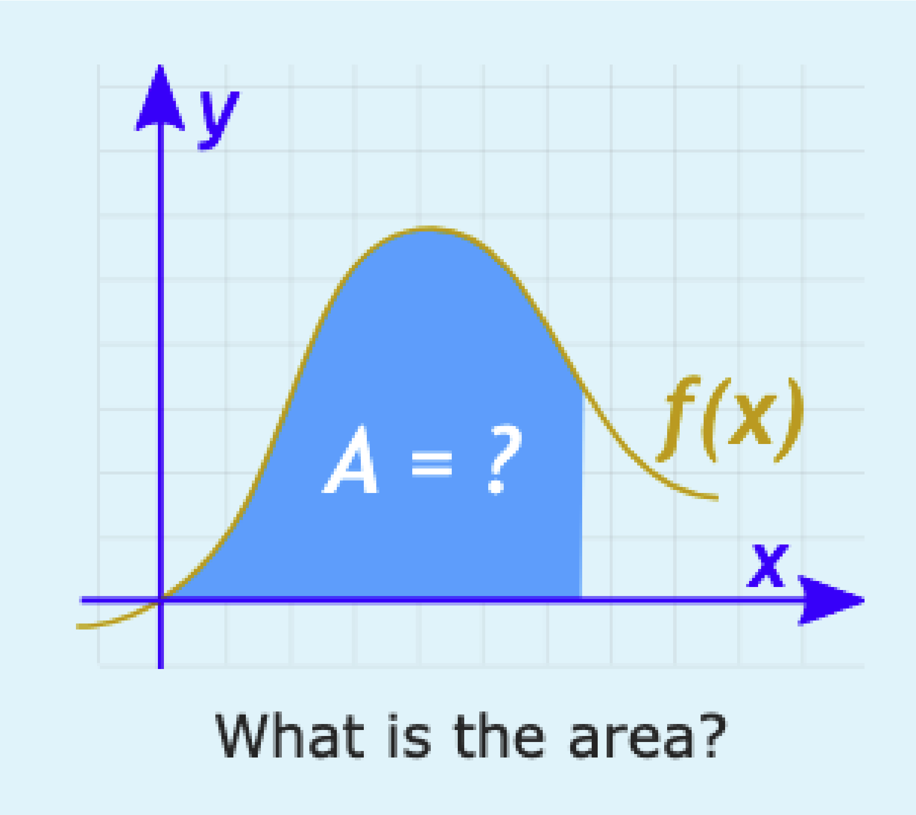
| Common Functions | Function | Derivative |
| --- | --- | --- |
| Constant | $c$ | $0$ |
| Line | $x$ | $1$ |
| | $ax$ | $a$ |
| Square | $x^2$ | $2x$ |
| Square Root | $\sqrt{x}$ | $(½)x^{-½}$ |
| Exponential | $e^x$ | $e^x$ |
| | $a^x$ | $\ln(a)\ a^x$ |
| Logarithms | $\ln(x)$ | $1/x$ |
| | $\log_a(x)$ | $1\ /\ (x \ln(a))$ |
| Trigonometry (x is in radians) | $\sin(x)$ | $\cos(x)$ |
| | $\cos(x)$ | $-\sin(x)$ |
| | $\tan(x)$ | $\sec^2(x)$ |
| Inverse Trigonometry | $\sin^{-1}(x)$ | $1/\sqrt{(1-x^2)}$ |
| | $\cos^{-1}(x)$ | $-1/\sqrt{(1-x^2)}$ |
| | $\tan^{-1}(x)$ | $1/(1+x^2)$ |

| Rules | Function | Derivative |
| --- | --- | --- |
| Multiplication by constant | $cf$ | $cf'$ |
| Power Rule | $x^n$ | $nx^{n-1}$ |
| Sum Rule | $f + g$ | $f' + g'$ |
| Difference Rule | $f - g$ | $f' - g'$ |
| Product Rule | $fg$ | $f\,g' + f'\,g$ |
| Quotient Rule | $f/g$ | $\frac{f'\,g - g'\,f}{g^2}$ |
| Reciprocal Rule | $1/f$ | $-f'/f^2$ |
| | | |
| Chain Rule (using ′ ) | $f(g(x))$ | $f'(g(x))g'(x)$ |
| Chain Rule (as "Composition of Functions") | $f \circ g$ | $(f' \circ g) \times g'$ |
| Chain Rule (using $\frac{d}{dx}$ ) | $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ | |

Source: mathisfun.com

# Integral

*The <u>integral</u> computes the area under a function…*

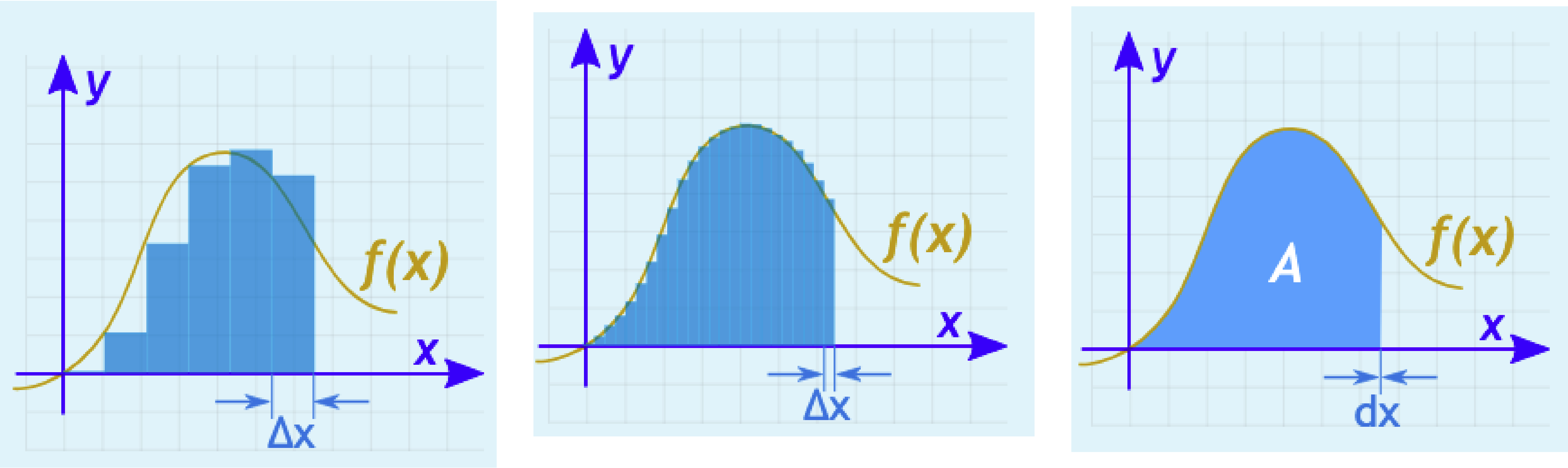



*…it is also the reverse operation of a derivative.*

Source: mathisfun.com

# Integration

*We can view this as adding up smaller and smaller slices…*




*…where dx means slices are approaching zero width.*

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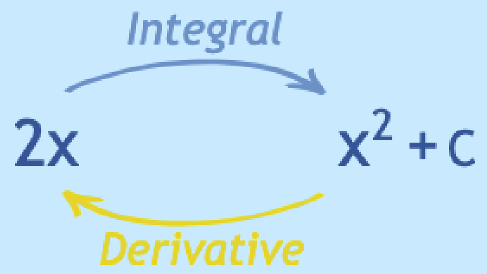
# Integration

*But there is a shortcut because the integral is the reverse of the derivative…*

Example: $2x$

An integral of $2x$ is $x^2$ ...




... because the derivative of $x^2$ is $2x$

(More about "+C" later.)

Source: mathisfun.com

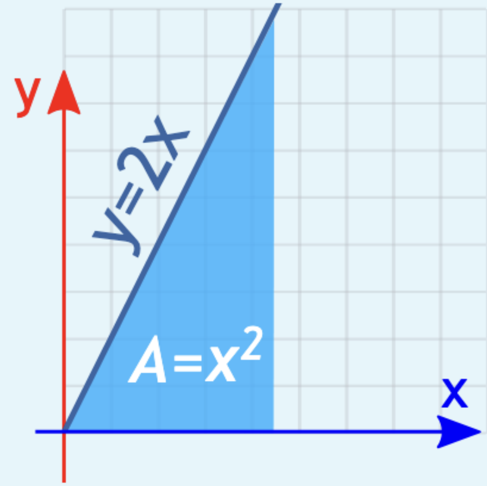
# Integral Notation

Slices along x

$$\int 2x\,dx = x^2 + C$$

Integral Symbol

Function we want to integrate




Area of triangle $= \frac{1}{2}$(base)(height) $= \frac{1}{2}(x)(2x) = x^2$

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# Plus C

$$\int 2x\,dx = x^2 + C$$

- "Constant of Integration"
- Captures all functions whose derivative is 2x
- Derivative of a constant C is 0 so…
  - Derivative of $x^2$ is 2x
  - Derivative of $x^2 + 4$ is 2x
  - Derivative of $x^2 + 99$ is 2x
  - Etc.
- Basically, just always add + C…

# Rules of Integration

| Common Functions | Function | Integral |
|---|---|---|
| Constant | $\int a\,dx$ | $ax + C$ |
| Variable | $\int x\,dx$ | $x^2/2 + C$ |
| Square | $\int x^2\,dx$ | $x^3/3 + C$ |
| Reciprocal | $\int (1/x)\,dx$ | $\ln\|x\| + C$ |
| Exponential | $\int e^x\,dx$ | $e^x + C$ |
| | $\int a^x\,dx$ | $a^x/\ln(a) + C$ |
| | $\int \ln(x)\,dx$ | $x \ln(x) - x + C$ |
| Trigonometry (x in radians) | $\int \cos(x)\,dx$ | $\sin(x) + C$ |
| | $\int \sin(x)\,dx$ | $-\cos(x) + C$ |
| | $\int \sec^2(x)\,dx$ | $\tan(x) + C$ |
| | | |
| **Rules** | **Function** | **Integral** |
| Multiplication by constant | $\int cf(x)\,dx$ | $c\int f(x)\,dx$ |
| Power Rule ($n \neq -1$) | $\int x^n\,dx$ | $\frac{x^{n+1}}{n+1} + C$ |
| Sum Rule | $\int (f + g)\,dx$ | $\int f\,dx + \int g\,dx$ |
| Difference Rule | $\int (f - g)\,dx$ | $\int f\,dx - \int g\,dx$ |
| Integration by Parts | See Integration by Parts | |
| Substitution Rule | See Integration by Substitution | |

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# Example: Power Rule

Example: What is $\int x^3\,dx$ ?

The question is asking "what is the integral of $x^3$ ?"

We can use the Power Rule, where n=3:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^3\,dx = \frac{x^4}{4} + C$$

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# Example: Power Rule

**Example: What is $\int\sqrt{x}\,dx$ ?**

$\sqrt{x}$ is also $\mathbf{x^{0.5}}$

We can use the Power Rule, where n=0.5:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^{0.5}\,dx = \frac{x^{1.5}}{1.5} + C$$

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# Example: Multiplication by a Constant

**Example: What is $\int 6x^2\,dx$ ?**

We can move the 6 outside the integral:

$$\int 6x^2\,dx = 6\int x^2\,dx$$

And now use the Power Rule on $x^2$:

$$= 6\,\frac{x^3}{3} + C$$

Simplify:

$$= 2x^3 + C$$

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# Definite Integral

*Same concept as a discrete summation…*

$$\sum_{x=a}^{b} 2x$$

*Discrete*

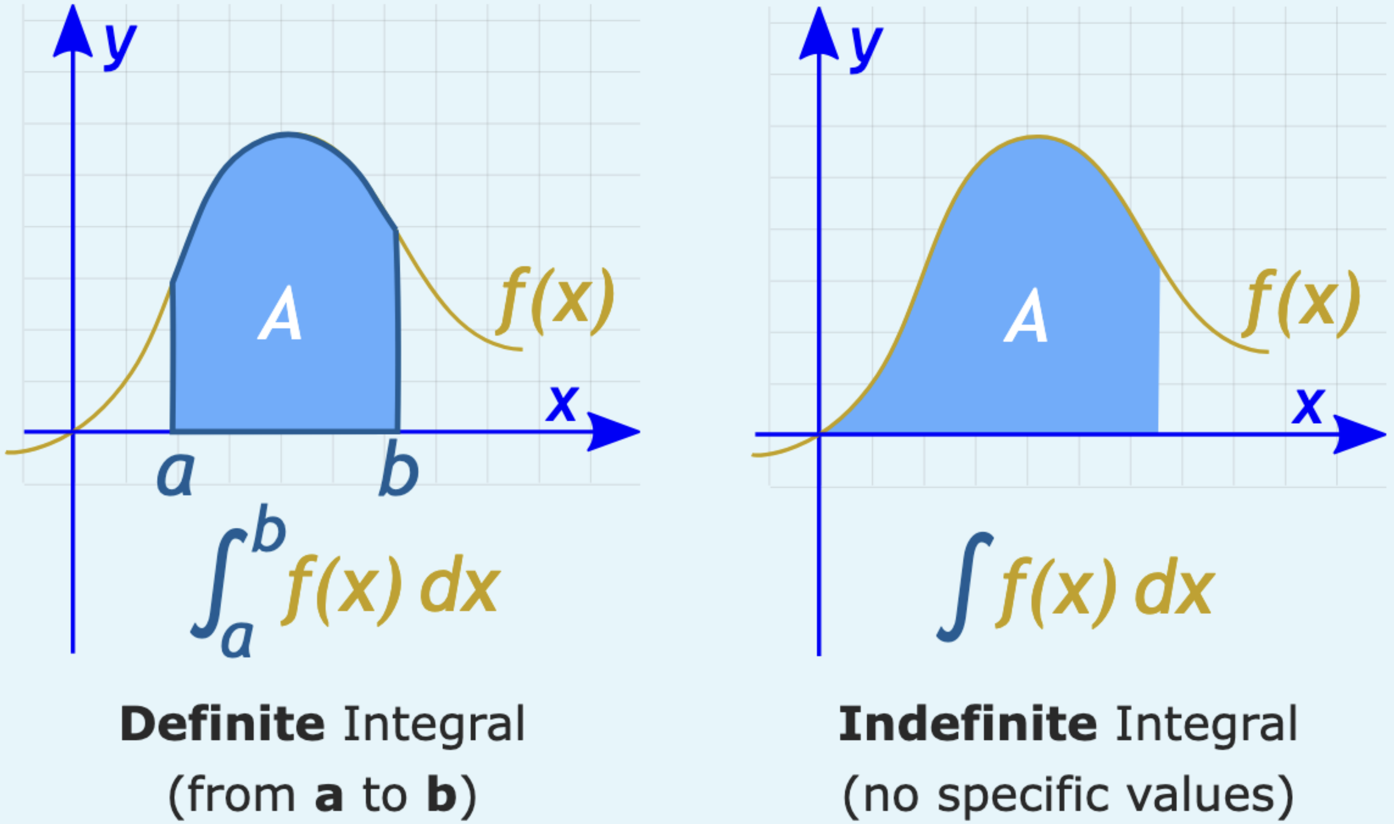
$$\int_{a}^{b} 2x \, dx$$

*Continuous*

*…integral symbol is a stylized "S" for "sum".*

# Definite Integral

***Definite Integral** can be used to find the area under a curve between two points…*

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# Definite Integral

First we compute the indefinite integral,

$$\int_a^b 2x\,dx = x^2 + C$$

Then we substitute and subtract the endpoints,

$$\left[x^2 + C\right]_a^b = b^2 - a^2$$

Notice that the constant of integration always cancels,

$$\left[x^2 + C\right]_a^b = (b^2 + C) - (a^2 + C) = b^2 - a^2$$

# Example: Definite Integral

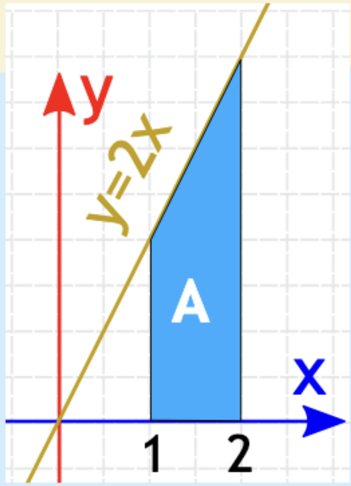
Example (continued)

How to show your answer:

$$\int_1^2 2x\,dx = \left[\, x^2 \,\right]_1^2$$

$$= 2^2 - 1^2$$

$$= \mathbf{3}$$



# Outline

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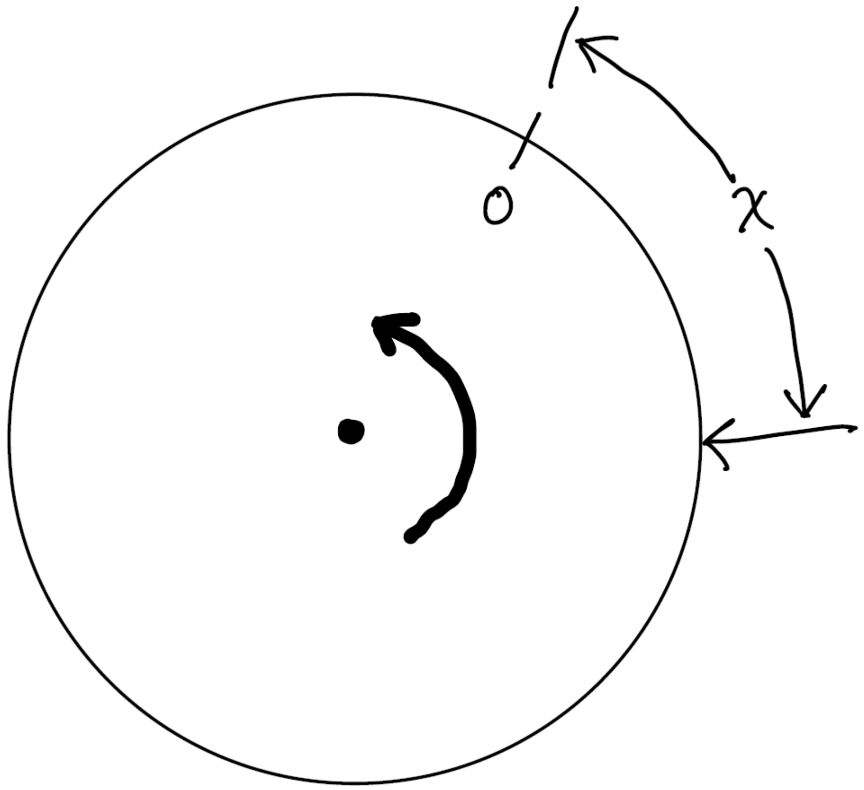
- Continuous Probability Distributions
- Fundamental Rules of Probability

# Continuous Probability

**Experiment** Spin continuous wheel and measure X displacement from 0




**Question** Assuming uniform probability, what is $P(X = x)$?

# Continuous Probability

For continuous random variables, the probability of any outcome is zero:

$$P(X = x) = 0$$

Our preference is to define events as intervals, e.g. using the CDF:

$$P(X \leq x), \quad P(a < X \leq b), \quad P(X > c), \quad \text{etc.}$$

Note that when X is continuous,

$$P(a < X \leq b) = P(a < X < b) + P(X = b) = P(a < X < b)$$

thus, it does not matter if we include the endpoints.

# Probability Density

**Definition** A function $p(X)$ is a **probability density function (PDF)** of a continuous random variable X if the following hold:

(a) It is nonnegative for all values in the support,

$$p(X = x) \geq 0$$

(b) The integral over all values in the support is 1,

$$\int p(X = x)\, dx = 1$$

(c) The probability of an interval is given by the integral of the PDF,

$$P(a < X < b) = \int_a^b p(X = x)\, dx$$

# Continuous Probability

$$P(a \le X < b) = \int_a^b p(X = x)\, dx$$

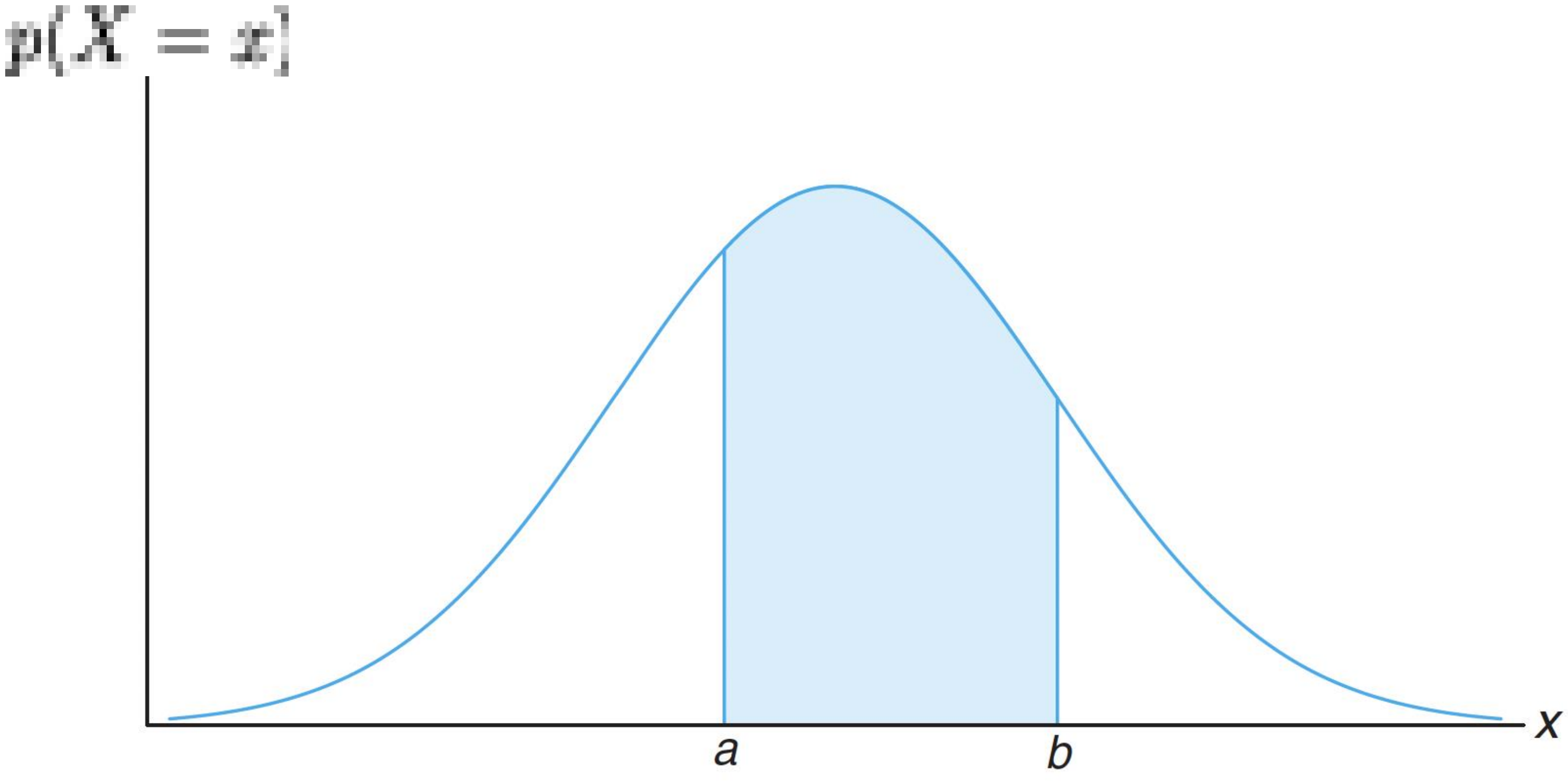



Figure 3.5: $P(a < X < b)$.

# Continuous Probability Measures

- Can easily measure probability of closed intervals,

$$P(a \leq X < b) = P(X < b) - P(X < a)$$

- If *X* is *absolutely continuous* (i.e. differentiable) then,

Fundamental Theorem of Calculus

$$p(x) = \frac{dP(x)}{dx} \quad \text{and} \quad P(t) = \int_{-\infty}^{t} p(x)\, dx$$

Where $p(x)$ is the *probability density function* (PDF)

# A word on notation…

*The book uses slightly different notation from my slides…*

The function $f(x)$ indicates the PMF or PDF:

$f(x) = P(X = x)$ for discrete X, and $f(x) = p(X = x)$ for continuous X.

The function $F(x)$ denotes the CDF for discrete and continuous RVs X:

$$F(x) = P(X < x)$$

*…I will largely avoid the $f(x)$ and $F(x)$ notation except for some examples.*

# Outline

- Concepts of Calculus
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# Continuous Probability

*Most fundamental rules hold, replacing PMF with PDF/CDF…*

Two RVs X & Y are **independent** if and only if,

$$p(x, y) = p(x)p(y) \quad \text{or} \quad P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

**Conditionally independent** given Z iff,

**Shorthand:** $P(x) = P(X \leq x)$

$$p(x, y \mid z) = p(x \mid z)p(y \mid z) \quad \text{or} \quad P(x, y \mid z) = P(x \mid z)P(y \mid z)$$

**Probability chain rule**,

$$p(x, y) = p(x)p(y \mid x) \quad \text{and} \quad P(x, y) = P(x)P(y \mid x)$$

# Continuous Probability

*…and by replacing summation with integration…*

**Law of Total Probability** for continuous distributions,

$$p(x) = \int_{\mathcal{Y}} p(x, y)\, dy$$

**Conditional density** of a continuous random variable,

$$p(x \mid y) = \frac{p(x, y)}{p(y)}$$

Combining these we have the following:

$$p(x \mid y) = \frac{p(x, y)}{\int_{\mathcal{X}} p(x, y)\, dx}$$

# Continuous Probability

***Caution*** *Some technical subtleties arise in continuous spaces…*

For **discrete** RVs X & Y, the conditional

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

is **undefined** when P(Y=y) = 0 … no problem.

For **continuous** RVs we have,

$$P(X \leq x \mid Y = y) = \frac{P(X \leq x, Y = y)}{P(Y = y)}$$

but numerator and denominator are 0/0.

*…we will just work with the conditional PDF for now…*

# Fundamental Rules of Probability

**Law of total probability**

$$p(Y) = \int p(Y, X = x)\, dx$$

- p(y) is a **marginal** distribution
- This is called **marginalization**

**Proof**

$$\int p(Y, X = x)\, dx = \int p(Y) p(X = x \mid Y)\, dx$$ ( chain rule )

$$= p(Y) \int p(X = x \mid Y)\, dx$$ ( distributive property )

$$= p(Y)$$ ( PDF Integrates to 1 )

# Fundamental Rules of Probability

Given two continuous RVs $X$ and $Y$ the **conditional density** is:

$$p(X \mid Y) = \frac{p(X,Y)}{p(Y)}$$

By the law of total probability, we also have the definition:

$$p(X \mid Y) = \frac{p(X,Y)}{\int p(X=x,Y)\,dx}$$

# Independence of RVs

**Definition** *Two random variables* $X$ *and* $Y$ *are* <u>*independent*</u> *if and only if,*

$$p(X = x, Y = y) = p(X = x)p(Y = y)$$

- This must hold for all values $x$ and $y$.
- If for any values $x$ and $y$,

$$p(X = x, Y = y) \neq p(X = x)p(Y = y)$$

  then X and Y are **dependent.**
- Example: Rolling two dice, each die is independent of the other
- Independence is *symmetric:* if X is independent of Y then Y is independent of X
- Equivalent definition of independence: $p(X \mid Y) = p(X)$

# Independence of RVs

**Definition** RVs $X_1, X_2, \ldots, X_N$ are mutually independent if and only if,

$$p(X_1 = x_1, \ldots, X_N = x_N) = \prod_{i=1}^{N} p(X_i = x_i)$$

*In words:* If a set of random variables is independent, then their joint probability is a product of their marginals.

# Independence of RVs

**Definition** *Two random variables* $X$ *and* $Y$ *are* <u>*conditionally independent*</u> *given* $Z$ *if and only if,*

$$p(X = x, Y = y \mid Z = z) = p(X = x \mid Z = z)p(Y = y \mid Z = z)$$

for all values $x$, $y$, and $z$.

- N RVs conditionally independent, given Z, if and only if:

$$p(X_1 = x_1, \ldots, X_N = X_N \mid Z = z) = \prod_{i=1}^{N} p(X_i = x_i \mid Z = z)$$

- Equivalent def'n of conditional independence: $p(X \mid Y, Z) = p(X \mid Z)$
- Conditional independence is symmetric